

# Anomaly Induced Transport in Arbitrary Dimensions

---

**R. Loganayagam\***

Junior Fellow, Harvard Society of Fellows,  
Harvard University, Cambridge, MA 02138 .

**ABSTRACT:** Motivated by the consistency of a global anomaly with the second law of thermodynamics, we propose a form for the anomaly induced charge/energy transport in arbitrary even dimensions. In a given dimension, this form exhausts all second law constraints on anomaly induced transport at any given order in hydrodynamic derivative expansion. We achieve by solving the second law constraints off-shell without resorting to hydrodynamic equations at lower orders. We also study various possible finite temperature corrections to such anomaly induced transport coefficients.

**KEYWORDS:** .

---

\*nayagam@physics.harvard.edu

---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Basic formalism</b>	<b>2</b>
<b>3. Anomalous transport in General dimensions</b>	<b>6</b>
<b>4. Comparison with Son-Surowka result</b>	<b>8</b>
<b>5. Discussion</b>	<b>10</b>
<b>A. Finite temperature corrections to anomaly-induced transport</b>	<b>11</b>
<b>B. Explicit expressions in various cases</b>	<b>13</b>
<b>C. Notation</b>	<b>17</b>

---

## 1. Introduction

One of the interesting features to come out of studying hydrodynamics of quantum field theories via holography [1, 2] is the discovery that there are macroscopic transport coefficients in the hydrodynamics which probe the global anomalies in the microscopic theory[3, 4, 5, 6]. Since the anomalies are exact objects in field theory, this leads to the hope that there are certain transport coefficients of interacting field theories which can be exactly computed by relating them to underlying anomalies.

Son and Surowka in [7] constrained various such first order transport coefficients using the consistency of the (3+1)d-anomaly with the second law of thermodynamics. Motivated by the form of their answer, we propose a generalization of the anomaly induced charge and energy transport in arbitrary dimensions and show that the form is consistent with the second law<sup>1</sup>.

We will argue that the simple form of anomaly induced transport presented in this paper exhausts all constraints on anomaly induced transport (at arbitrary order in hydrodynamic derivative expansion) that can ever be derived from second law considerations

---

<sup>1</sup>Recently, there have been indications that there might be additional transport [8] induced at finite temperature by the mixed gravitational anomalies[9, 10] in (3+1)dimensions at least in free theories. All these results generalize the results of various old calculations in the literature[11, 12, 13, 14, 15, 16]. However, a general argument linking these transport to mixed anomalies in an arbitrary interacting system is presently lacking (at least the author is unaware of any such argument in the literature).

In this article, we will neglect such additional contributions and focus only on purely global anomalies. However it is possible that the succinct forms of the global anomaly induced transport presented in this paper will lead to a suitable ansatz taking into account gravitational anomalies.

alone (in a flat spacetime). This is achieved by solving the second law constraints off-shell without resorting to hydrodynamic equations at lower orders in derivative expansion<sup>2</sup>.

The particular form of anomaly induced transport that we propose has various structural features and we resort to a specific notation which exhibits these features clearly. We begin by establishing the relevant notation/formalism in section §2. In particular, we introduce a useful notation which streamlines the entire discussion.

We present the advertised generalization in the section §3 and prove its consistency with the second law. In the next section §4 we restrict to (3+1)dimensions and show that under particular frame redefinition, the results in [7] are recovered. We conclude in §5 with the discussion of various issues and future directions.

In the appendix §A, we discuss possible finite temperature corrections to the anomaly-induced transport. These corrections represent other possible non-dissipative transport which are very similar in their structure to anomaly induced transport phenomena and are interesting in their own right.

In the next appendix §B, we present various expressions relevant for dimensions upto  $d = 10$  for ready reference. The last appendix §C is a summary of various notations employed in this paper.

*Note:* As this paper was readied for submission, a preprint addressing similar questions [17] appeared on arXiv.

## 2. Basic formalism

We are interested in hydrodynamics of field theories in arbitrary even spacetime dimensions. Let us denote the spacetime dimension by  $2n$ . We will assume that the field theory has a global symmetry which has a global anomaly - by this we mean, the covariant divergence of the covariant global current<sup>3</sup> is given by a series of terms involving the non-dynamical gauge fields associated with this global symmetry. This in turn means that we can use the standard ‘covariant anomaly’ in our discussion - with the anomaly symmetrically shared between the currents<sup>4</sup>.

---

<sup>2</sup>This of course leaves open the possibility that there are additional transport terms which are in general induced by the anomaly but are not captured by a second-law type argument. Such terms, if present, would be beyond the scope of this paper and their understanding would then crucially depend on finding a field theoretical way to derive these transport coefficients in general interacting QFTs.

<sup>3</sup>In general, the currents obtained by a direct variation of the path integral (the “consistent” currents) are *not* covariant under the shift of the non-dynamical gauge fields. The covariant currents are obtained by adding by hand an additional current contribution called the Bardeen-Zumino current [18]. Note that such ad-hoc additions are allowed in global symmetry currents and would have been illegitimate for any current coupled to a dynamical gauge field.

<sup>4</sup>Note that if we wanted to give dynamics to a subset of gauge fields perturbatively, then (assuming we start from a path-integral measure which gives the symmetric ‘consistent anomaly’) we have to add local counterterms (called Bardeen counterterms) made purely out of gauge fields in the microscopic action so that we shift the anomalies away from the gauged subgroup (A ready reference for Bardeen counterterms in arbitrary dimensions is [19]). Any counter-term can be thought of as a redefinition of the path-integral measure, and in this case we are just redefining the measure so that it is invariant under the gauged subgroup.

We want to study hydrodynamics at finite temperature and cartan chemical potentials for the global symmetry. We want to do this in the presence of the non-dynamical gauge fields also turned on in the same cartan subgroup<sup>5</sup>. When a global symmetry has an anomaly, the constitutive relations in the ideal hydrodynamics are modified - we will find it convenient to present the modifications in the following form for the energy, charge and the entropy currents<sup>6</sup>

$$\begin{aligned} T^{\mu\nu} &\equiv \varepsilon u^\mu u^\nu + p P^{\mu\nu} + q_{anom}^\mu u^\nu + u^\mu q_{anom}^\nu + T_{diss}^{\mu\nu} \\ J^{i\mu} &\equiv n^i u^\mu + J_{anom}^{i\mu} + J_{diss}^{i\mu} \\ J_S^\mu &\equiv s u^\mu + J_{S,anom}^\mu + J_{S,diss}^\mu \end{aligned} \quad (2.1)$$

where  $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ , pressure of the fluid is  $p$  and  $\{\varepsilon, n^i, s\}$  are the energy, charge and the entropy densities respectively.

$u^\mu$  is the velocity of the fluid under consideration which obeys  $u^\mu u_\mu = -1$ . We have denoted by  $\{q_{anom}^\mu, J_{anom}^{i\mu}, J_{S,anom}^\mu\}$  the anomalous heat/charge/entropy currents and by  $\{T_{diss}^{\mu\nu}, J_{diss}^{i\mu}, J_{S,diss}^\mu\}$  the dissipative currents. These are assumed to satisfy<sup>7</sup>

$$\begin{aligned} u_\mu q_{anom}^\mu &= u_\mu J_{anom}^{i\mu} = u_\mu J_{S,anom}^\mu = 0 \\ D_\mu J^{i\mu} &= \mathfrak{A}^i \\ D_\mu T^{\mu\nu} &= J_\mu^i F_i^{\nu\mu} \end{aligned} \quad (2.2)$$

where  $D$  is the covariant derivative including gauge fields and  $\mathfrak{A}^i$  is the global anomaly.

---

So, whenever a subgroup is gauged, all currents get corrected due to a Bardeen current coming from these additional terms which is a function of various gauge fields. These Bardeen counterterms depend on the details of the subgroup that is gauged and the effect of these terms on the final hydrodynamics depends further on the details of the quantum gauge dynamics at finite temperature/chemical potential.

The results derived here might be of some relevance to such mixed anomalies between global symmetries and gauge redundancies only if this gauge dynamics is perturbative, i.e., if the theory is ‘weakly gauged’ in which case the effect of Bardeen terms can be accounted for perturbatively.

<sup>5</sup>Note that these non-dynamical gauge fields violate the conservation of the global charges. Thus, by introducing chemical potentials for these charges, we are making an implicit assumption that this non-conservation is in some sense ‘small’ enough that it makes sense to talk about local thermal/chemical equilibrium within each fluid element.

<sup>6</sup>Note that we do not include any tensor corrections to the energy momentum tensor due to the anomalous transport. As will be seen later, the second law considerations do not constrain such terms and hence such terms do not fall into the ambit of the anomalous transport terms considered in this paper. It is of course possible that more microscopic considerations lead to additional transport phenomena which a second law argument is blind to. Further, in specific theories, anomalies might, in consonance with other features of the theory, lead to specific phenomena which cannot be captured by any such general analysis. All subsequent analysis is subject to these caveats. The author wishes to thank Jyotirmoy Bhattacharya for emphasizing this point.

<sup>7</sup>Note that we do not insist on particular frame conventions that need to be used for the dissipative parts of the constitutive relation. So, most of what we say would work in any frame *provided the frame convention does not explicitly forbid the forms in which anomalous pieces appear in our equation*. For example, one of the commonly employed frames is the Landau frame where terms like  $q_{anom}^{(\mu} u^{\nu)}$  are explicitly forbidden. To figure out the anomaly-induced transport in such frames, one needs to first shift to a frame which allows such terms, add the transport in the new frame and then shift back. The author wishes to thank Jyotirmoy Bhattacharya and Sayantani Bhattacharyya for discussions related to this point.

Now the rate of entropy production is given by

$$\begin{aligned}
TD_\mu J_S^\mu &= TD_\mu J_S^\mu + \mu_i [D_\mu J^{i\mu} - \mathfrak{A}^i] + u_\nu [D_\mu T^{\mu\nu} - J_\mu^i F_i^{\nu\mu}] \\
&= u^\mu [TD_\mu s + \mu_i D_\mu n^i - D_\mu \varepsilon] + D_\mu u^\mu [Ts + \mu_i n^i - (\varepsilon + p)] \\
&\quad + J_{anom}^{i\mu} E_{i\mu} - \mu_i \mathfrak{A}^i + TD_\mu J_{S,anom}^\mu + \mu_i D_\mu J_{anom}^{i\mu} - (D_\mu + a_\mu) q_{anom}^\mu \\
&\quad + TD_\mu J_{S,diss}^\mu + \mu_i D_\mu J_{diss}^{i\mu} + u_\nu [D_\mu T_{diss}^{\mu\nu} - J_{\mu,diss}^i F_i^{\nu\mu}] \\
&= J_{anom}^{i\mu} E_{i\mu} - \mu_i \mathfrak{A}^i + TD_\mu J_{S,anom}^\mu + \mu_i D_\mu J_{anom}^{i\mu} - (D_\mu + a_\mu) q_{anom}^\mu \\
&\quad + TD_\mu J_{S,diss}^\mu + \mu_i D_\mu J_{diss}^{i\mu} + u_\nu [D_\mu T_{diss}^{\mu\nu} - J_{\mu,diss}^i F_i^{\nu\mu}]
\end{aligned} \tag{2.3}$$

where  $E_{i\mu} \equiv F_{i\mu\nu} u^\nu$  is the rest frame electric field and  $a_\mu \equiv (u.D)u_\mu$  is the acceleration field.

The second law of thermodynamics is the assertion that the RHS of the above equation be positive. If we assume that the anomalous transport terms do not contribute to the entropy production then we get the standard second law constraint on the dissipative part when there is no anomaly, viz., assuming

$$(D_\mu + a_\mu) q_{anom}^\mu - J_{anom}^{i\mu} E_{i\mu} = TD_\mu J_{S,anom}^\mu + \mu_i [D_\mu J_{anom}^{i\mu} - \mathfrak{A}^i] \tag{2.4}$$

we get the second law constraint as

$$TD_\mu J_{S,diss}^\mu + \mu_i D_\mu J_{diss}^{i\mu} + u_\nu [D_\mu T_{diss}^{\mu\nu} - J_{\mu,diss}^i F_i^{\nu\mu}] \geq 0 \tag{2.5}$$

Note that if we can find even one solution that satisfies equation(2.4), we can use that to reduce the second-law to the usual non-anomalous case .

To be useful this way the solution should be an ‘off-shell’ solution i.e., we should be able to show that it is a solution *without any further use of the conservation equations* which involve dissipative contributions from various orders<sup>8</sup>. The main result of this paper is to demonstrate by constructing a solution that this can always be done in arbitrary dimensions<sup>9</sup>.

It is convenient to write the equation(2.4) in terms of forms. To this end, introduce the hodge dual  $2n - 1$  forms  $\{\bar{J}_{S,anom}, \bar{q}_{anom}, \bar{J}_{anom}^i\}$ . We remind the reader that in  $2n$  dimensions, given any  $2n - 1$  form  $\bar{V}$  hodge-dual to  $V_\mu$  and a 1-form  $A_\mu$ , we have

$$\begin{aligned}
D\bar{V} &= (D_\mu V^\mu) \text{Vol}_{2n} \\
A \wedge \bar{V} &= -\bar{V} \wedge A = A_\mu V^\mu \text{Vol}_{2n}
\end{aligned} \tag{2.6}$$

So the equation(2.4) can be recast as

$$TD\bar{J}_{S,anom} = D\bar{q}_{anom} + a \wedge \bar{q}_{anom} + \bar{J}_{anom}^i \wedge E_i - \mu_i [D\bar{J}_{anom}^i - \bar{\mathfrak{A}}^i] \tag{2.7}$$

---

<sup>8</sup>In particular, this is not true of the solution presented by Son and Surowka in [7] since they use the ideal fluid equations of motion.

<sup>9</sup>Note that if we succeed in finding one such solution without any entropy production we would have also demonstrated by construction that there is a piece of transport linking second law and anomalies which is non-dissipative. Hence, our assumption about the non-dissipative nature of anomalous transport would be justified in hindsight

where  $\bar{\mathfrak{A}}^i \equiv \mathfrak{A}^i \text{Vol}_{2n}$  is the  $2n$ -form Hodge dual to the 0-form  $\mathfrak{A}^i$ .

Now, we turn to a more detailed analysis of the anomaly. Let  $F_i$  be the field-strength 2-form and we have already defined the electric 1-form via  $E_{i\mu} \equiv F_{i\mu\nu}u^\nu$ . We can do an electric-magnetic decomposition

$$F_{i\mu\nu} - [u_\mu E_{i\nu} - E_{i\mu}u_\nu] \equiv B_{i\mu\nu} \quad (2.8)$$

or in the language of forms

$$F_i = B_i + u \wedge E_i \quad (2.9)$$

where  $B_i$  is the magnetic 2-form completely transverse to  $u^\mu$ , i.e.,  $B_{i\mu\nu}u^\nu = 0$ . In fact if our spacetime dimension is  $2n$ , in the rest frame of the fluid  $B_i$  can be thought of as 2-form in  $2n - 1$  spatial dimensions. This in particular means that

$$B_{i_1} \wedge B_{i_2} \wedge B_{i_3} \dots \wedge B_{i_n} = 0 \quad (2.10)$$

where  $i_k$ s are the flavor indices.

We will also use the standard decomposition of the velocity gradients

$$D_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} - u_\mu a_\nu + \frac{\theta}{d-1} P_{\mu\nu} \quad (2.11)$$

in terms of the shear strain rate  $\sigma_{\mu\nu}$ , the vorticity  $\omega_{\mu\nu}$ , the acceleration  $a_\mu$  and the expansion rate  $\theta$  of the fluid. This in particular means the exterior derivative of the velocity 1-form has the decomposition

$$Du = 2\omega - u \wedge a \quad (2.12)$$

where  $\omega$  is the vorticity 2-form. Further, using  $DF_i = D^2u = 0$ , we get

$$\begin{aligned} DB_i &= DF_i - Du \wedge E_i + u \wedge DE_i \\ &= -Du \wedge E_i + u \wedge DE_i \\ &= -2\omega \wedge E_i + u \wedge (DE_i + a \wedge E_i) \\ 2D\omega &= Du \wedge a - u \wedge Da = 2\omega \wedge a - u \wedge Da \\ D(B_i + 2\mu_i\omega) &= -2\omega \wedge (E_i - D\mu_i - a\mu_i) + u \wedge (DE_i + a \wedge E_i - Da) \end{aligned} \quad (2.13)$$

We can write the anomaly in  $2n$  dimensions in the form

$$\begin{aligned} \bar{\mathfrak{A}}^{i_0} &= \frac{1}{n!} \mathfrak{C}^{i_0 i_1 i_2 i_3 \dots i_n} F_{i_1} \wedge F_{i_2} \wedge F_{i_3} \dots \wedge F_{i_n} \\ &= \frac{1}{(n-1)!} \mathfrak{C}^{i_0 i_1 i_2 i_3 \dots i_n} B_{i_1} \wedge B_{i_2} \wedge B_{i_3} \dots \wedge B_{i_{n-1}} \wedge u \wedge E_{i_n} \end{aligned} \quad (2.14)$$

where  $\mathfrak{C}^{i_0 i_1 i_2 i_3 \dots i_n}$  is the anomaly coefficient which is completely symmetric in all its indices.

All our subsequent expressions will be linear in the anomaly coefficient. We will exploit this by introducing a notational trick which simplifies the presentation. We will begin by doing a replacement

$$\mathfrak{C}^{i_0 i_1 \dots i_n} \mapsto t^{i_0} t^{i_1} \dots t^{i_n}$$

where we have replaced the anomaly coefficient with a bunch of fictitious parameters. This turns the above equation to

$$\bar{\mathfrak{A}}^i = t^i \frac{(t.F)^n}{n!} = t^i \frac{(t.B)^{n-1}}{(n-1)!} \wedge u \wedge (t.E) \quad (2.15)$$

we can always restore the original expression by doing the reverse replacement

$$t^{i_0} t^{i_1} \dots t^{i_n} \mapsto \mathfrak{C}^{i_0 i_1 \dots i_n}$$

We will further find it convenient to do a formal sum over all the even dimensions, writing<sup>10</sup>

$$\bar{\mathfrak{A}}^i = t^i e^{t.F} = t^i e^{t.B} \wedge u \wedge (t.E) \quad (2.16)$$

Hence, in the following, to obtain the formulae for  $2n$  dimensions we have to Taylor-expand in  $t^i$ 's, take the terms with  $n+1$  number of  $t^i$ 's and do the replacement

$$t^{i_0} t^{i_1} \dots t^{i_n} \mapsto \mathfrak{C}^{i_0 i_1 \dots i_n}$$

where  $\mathfrak{C}^{i_0 i_1 \dots i_n}$  is the anomaly-coefficient of the theory.

### 3. Anomalous transport in General dimensions

We now want to present a solution for the equation(2.7) which we repeat here

$$TD\bar{J}_{S,anom} + \mu_i [D\bar{J}_{anom}^i - \bar{\mathfrak{A}}^i] = D\bar{q}_{anom} + a \wedge \bar{q}_{anom} + \bar{J}_{anom}^i \wedge E_i \quad (3.1)$$

We remind the reader that this is physically the statement that the anomalous transport leads to no entropy production. One simple solution of the above equation valid in arbitrary dimensions is<sup>11</sup>

$$\begin{aligned} \bar{q}_{anom} &= - \left[ \frac{(2t.\mu \omega - 1)e^{t.(B+2\mu\omega)} + e^{t.B}}{4\omega^2} \right] \wedge u \\ \bar{J}_{anom}^i &= -t^i \left[ \frac{e^{t.(B+2\mu\omega)} - e^{t.B}}{2\omega} \right] \wedge u \\ \bar{J}_{S,anom} &= 0 \end{aligned} \quad (3.2)$$

where we have used the notation introduced in the previous section to present the solution in arbitrary dimensions in a nice succinct form. This is the central result of this paper and we

---

<sup>10</sup>Note that we are denoting the formal sum over arbitrary dimensions by the same symbols as those used to denote quantities in a specific dimension. This is analogous to the notation used when say we want to represent anomaly polynomials by formal sum over all even dimensions. We hope the reader does not find this too confusing - our main motivation for this notation is that we believe it more clearly exhibits various structures associated with the anomaly-induced transport.

<sup>11</sup>Note that while these solutions as presented have  $\omega$  in their denominators, they have a smooth  $\omega \rightarrow 0$  limit.

will now demonstrate that this is indeed a solution. In our notation, this is straightforward . Using the results in equation(2.13), we have

$$\begin{aligned}
& D\bar{q}_{anom} + a \wedge \bar{q}_{anom} \\
&= \left\{ \frac{[(2t.\mu \omega - 1)t.(E - D\mu - a\mu) - (D\mu + a\mu)] e^{t.(B+2\mu\omega)} + (t.E)e^{t.B}}{2\omega} \right\} \wedge u \\
&\quad - \left[ \frac{(2t.\mu \omega - 1)e^{t.(B+2\mu\omega)} + e^{t.B}}{2\omega} \right]
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
& D\bar{J}_{anom}^i \\
&= t^i \left[ t.(E - D\mu - a\mu) e^{t.(B+2\mu\omega)} - (t.E)e^{t.B} \right] \wedge u \\
&\quad - t^i \left[ e^{t.(B+2\mu\omega)} - e^{t.B} \right]
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
& \mu_i [D\bar{J}_{anom}^i - t^i e^{t.B} \wedge u \wedge (t.E)] - [D\bar{q}_{anom} + a \wedge \bar{q}_{anom} + \bar{J}_{anom}^i \wedge E_i] \\
&= - \left[ \frac{e^{t.(B+2\mu\omega)} - (2t.\mu \omega + 1)e^{t.B}}{2\omega} \right]
\end{aligned} \tag{3.5}$$

Note that in the last line every term is a  $2n$  form in  $2n$  dimensions made only out of the ‘spatial’ forms  $\omega$  and  $B_i$  and any such form vanishes. Hence, we conclude that this is indeed the required ‘offshell’ solution.

Further it is interesting to note that our solution satisfies a First law-type relation

$$\frac{\partial \bar{q}_{anom}}{\partial \mu_i} = \mu_k \frac{\partial \bar{J}_{anom}^k}{\partial \mu_i} = -t^i (t.\mu) \left[ e^{t.(B+2\mu\omega)} \right] \wedge u \tag{3.6}$$

and a generalized Onsager- type Reciprocity relation

$$\frac{\delta \bar{q}_{anom}}{\delta B_i} = \frac{1}{2} \frac{\delta \bar{J}_{anom}^i}{\delta \omega} = -t^i \left[ \frac{(2t.\mu \omega - 1)e^{t.(B+2\mu\omega)} + e^{t.B}}{4\omega^2} \right] \wedge u \tag{3.7}$$

This means we can think of both the heat and the charge current as generated from a formal object  $\mathcal{V}_{anom}$

$$\begin{aligned}
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u \\
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u \\
\mathcal{V}_{anom} &\equiv - \left[ \frac{e^{t.(B+2\mu\omega)} - e^{t.B}}{2\omega} \right]
\end{aligned} \tag{3.8}$$

Note that  $\mathcal{V}_{anom}$  is a formal spatial  $2n$ -form in  $2n$  dimensions whose role is similar to the anomaly polynomial which is a formal  $2n + 2$  form in  $2n$  dimensions.



Further, we can think of these currents as derived from a single expression for the anomaly induced Gibbs free energy current  $\bar{\mathcal{G}}_{anom}$

$$\begin{aligned}
\bar{\mathcal{G}}_{anom} &= \left[ \frac{e^{t.(B+2\mu\omega)} - e^{t.B}(2t.\mu\omega + 1)}{(2\omega)^2} \right] = -\frac{1}{2\omega} [\mathcal{V}_{anom} - \mathcal{V}_{anom}(\omega = 0)] \wedge u \\
\bar{J}_{anom}^i &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial \mu_i} \\
\bar{J}_{S,anom} &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial T} \\
\bar{q}_{anom} &= \bar{\mathcal{G}}_{anom} + T \bar{J}_{S,anom} + \mu_i \bar{J}_{anom}^i
\end{aligned} \tag{3.9}$$

#### 4. Comparison with Son-Surowka result

In  $d = 4$  case (when  $n = 2$ ) we get the following result for the anomalous transport

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \frac{1}{2!} \mathfrak{C}^{ijk} F_j \wedge F_k \\
\bar{q}_{anom} &= -\mathfrak{C}^{ijk} \mu_i \mu_j \left[ \frac{1}{2} B_k + \frac{2}{3} \mu_k \omega \right] \wedge u \\
\bar{J}_{anom}^i &= -\mathfrak{C}^{ijk} \mu_j [B_k + \mu_k \omega] \wedge u \\
\bar{J}_{S,anom} &= 0
\end{aligned} \tag{4.1}$$

we take the Hodge-duals to get

$$\begin{aligned}
\mathfrak{A}^i &= \frac{1}{(2!)^3} \mathfrak{C}^{ijk} \epsilon^{\mu\nu\lambda\sigma} F_{j\mu\nu} \wedge F_{k\lambda\sigma} \\
q_{anom}^\mu &= -\frac{1}{2!} \mathfrak{C}^{ijk} \mu_i \mu_j \epsilon^{\mu\nu\lambda\sigma} \left[ \frac{1}{2} B_k + \frac{2}{3} \mu_k \omega \right]_{\nu\lambda} u_\sigma \\
J_{anom}^{i\mu} &= -\frac{1}{2!} \mathfrak{C}^{ijk} \mu_j \epsilon^{\mu\nu\lambda\sigma} [B_k + \mu_k \omega]_{\nu\lambda} u_\sigma \\
J_{S,anom}^\mu &= 0
\end{aligned} \tag{4.2}$$

To compare with [7], we first define

$$\mathfrak{C}^{ijk} \equiv -C^{ijk}, \quad \bar{B}_i^\mu \equiv \frac{1}{2!} \epsilon^{\mu\nu\lambda\sigma} B_{i\nu\lambda} u_\sigma \quad \text{and} \quad \bar{\omega}^\mu \equiv \frac{1}{2!} \epsilon^{\mu\nu\lambda\sigma} \omega_{\nu\lambda} u_\sigma$$

to get

$$\begin{aligned}
\mathfrak{A}^i &= -\frac{1}{8} C^{ijk} \epsilon^{\mu\nu\lambda\sigma} F_{j\mu\nu} \wedge F_{k\lambda\sigma} \\
q_{anom}^\mu &= C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] \\
J_{anom}^{i\mu} &= C^{ijk} \mu_j [\bar{B}_k^\mu + \mu_k \bar{\omega}^\mu] \\
J_{S,anom}^\mu &= 0
\end{aligned} \tag{4.3}$$

Son and Surowka presented their results in the Landau frame (where  $q^\mu = 0$ ) to first order in derivative expansion. The most general frame-change at this order leads to

$$\begin{aligned}
u^\mu &\mapsto u^\mu + \delta u^\mu + \dots \\
q^\mu &\mapsto q^\mu + (\varepsilon + p)\delta u^\mu + \dots \\
J^{i\mu} &\mapsto J^{i\mu} + n^i \delta u^\mu + \dots \\
J_S^\mu &\mapsto J_S^\mu + s \delta u^\mu + \dots
\end{aligned} \tag{4.4}$$

Hence to set  $q^\mu = 0$  we choose  $\delta u^\mu = -\frac{q^\mu}{\varepsilon + p}$  which gives

$$\begin{aligned}
\mathfrak{A}^i &= -\frac{1}{8} C^{ijk} \epsilon^{\mu\nu\lambda\sigma} F_{j\mu\nu} \wedge F_{k\lambda\sigma} \\
q_{anom}^\mu &= 0 \\
J_{anom}^{i\mu} &= C^{ijk} \mu_j [\bar{B}_k^\mu + \mu_k \bar{\omega}^\mu] - \frac{n^i}{\varepsilon + p} C^{ljk} \mu_l \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] + \dots \\
J_{S,anom}^\mu &= -\frac{s}{\varepsilon + p} C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] + \dots \\
&= -\frac{1}{T} C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] \\
&\quad + \frac{\mu_i n^i}{T(\varepsilon + p)} C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] + \dots \\
&= -\frac{1}{T} C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{2}{3} \mu_k \bar{\omega}^\mu \right] \\
&\quad - \frac{\mu_i J_{anom}^{i\mu}}{T} + \frac{1}{T} C^{ijk} \mu_i \mu_j [\bar{B}_k^\mu + \mu_k \bar{\omega}^\mu] + \dots \\
&= \frac{1}{T} C^{ijk} \mu_i \mu_j \left[ \frac{1}{2} \bar{B}_k^\mu + \frac{1}{3} \mu_k \bar{\omega}^\mu \right] - \frac{\mu_i J_{anom}^{i\mu}}{T} + \dots
\end{aligned} \tag{4.5}$$

which is exactly the expression obtained in [7].

Certain comments are in order - as the manipulations which lead to the above expressions make it clear, the expressions above get corrected at the next order in derivative expansion and in all subsequent orders. To find these corrections in the Landau frame, one needs to know the higher derivative pieces in the constitutive relation and the resultant corrections to the equations of motion. - hence, this is an ‘on-shell’ solution unlike the ‘off-shell’ solution we started with <sup>12</sup>. The beauty of the solution presented in our frame is that the solution is independent of all such detailed transport coefficients at lower orders in derivative expansion.

This exercise can be repeated in arbitrary dimensions in the Landau frame and the conclusions are similar. Whereas the leading order pieces in the anomalous transport can be determined via leading order equations of motion, the subleading orders require the knowledge of more pieces in the constitutive relation. This difficulty is again an artifact

---

<sup>12</sup>This statement holds true even for the additional terms proportional to temperature proposed in [8]. See appendix A for a more detailed discussion of such extra terms in arbitrary dimensions.

of working in the Landau frame : our solution gives a way of relaxing the second law constraints back to the non-anomalous case in most other frames. Given the complete constitutive relation, one can always frame-transform our solution to the Landau frame to get the anomalous transport to any order one wants.

## 5. Discussion

In this paper, we have proposed a form for the anomaly-induced transport in arbitrary even dimensions motivated by consistency with the second law. This transport takes a very suggestive form and it is tempting to speculate that this is related to some sort of an index-like object, especially since the anomalies themselves are computed by various indices.

We will present a particular suggestion for what this index might be. In presence of anomaly, the rotational equilibrium states get deformed because of the anomaly-induced transport. By taking a thermal density matrix of free chiral fermions on  $S^{2n-1} \times R$  with chemical potentials for angular momentum/charges turned on, we should be able to see the transport proposed in this paper<sup>13</sup>. This suggests that we are looking for a related index of some bundle on  $S^{2n-1} \times S^1$  with various appropriate twists. It will be nice if we could make this proposal precise. By fluid-gravity correspondence, similar rotational equilibrium states in large N strongly coupled theories are dual to blackhole solutions of Einstein-Maxwell-Chern-Simons actions in global  $AdS_{2n+1}$ . It is an interesting question whether these gravity solutions can be constructed exactly<sup>14</sup>.

As we have mentioned at various points in the paper, we have completely ignored the issue of pure and mixed gravitational anomalies in our discussion. Hopefully, the simple solution given in this paper and the methods used to construct it can be generalized to gravitational anomalies. The structure of various finite temperature corrections (discussed in Appendix §A ) and the 3 + 1d free-theory computation of [10] suggest relations between gravitational anomalies and various finite temperature corrections, but there is no definitive argument (apart from the statement that they have similar symmetry structure) to link these two. It would be instructive to see whether for free theories in arbitrary dimensions we still have a relation between finite temperature corrections to anomaly-induced transport on one hand and the coefficients of anomalies involving gravity on the other hand<sup>15</sup>. It is also worthwhile to independently understand these finite temperature corrections in various theories, as some of them violate CP and hence might of some phenomenological relevance.

The discovery of anomalies and their various effects have led us to a deeper understanding of QFTs with Lorentz-invariant ground states. While various effects of anomalies in finite temperature and finite density have been explored by now, we still lack a complete framework to encompass various anomaly-induced finite temperature/density transport. We hope that this work is a small step towards that broader goal.

---

<sup>13</sup>Note that this is a very similar computation to the ones performed by Vilenkin [11, 12, 13, 14, 15, 16] in flat-space in (3+1)d.

<sup>14</sup>The author wishes to thank Mukund Rangamani for related discussions.

<sup>15</sup>This for example might be achievable by computing these coefficients in kinetic theory of chiral fermion gases in arbitrary dimensions [20].

## Acknowledgements

It is a pleasure to thank Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Sean Hartnoll, Nabil Iqbal, Tongyan Lin, John Mason, Vladimir Manucharyan, Shiraz Minwalla, Mukund Rangamani, David Simmons-Duffin, Dam Thanh Son and Piotr Surowka for various useful discussions on ideas presented in this paper. I would like to thank ICTS, TIFR for their hospitality during the workshop on string theory and its applications organised at TIFR Mumbai. This work was supported by the Harvard Society of Fellows through a junior fellowship. Finally, I would like to thank various colleagues at the society for interesting discussions.

## Appendices

### A. Finite temperature corrections to anomaly-induced transport

In the main text of this article we presented a solution to the equation (2.7)

$$TD\bar{J}_{S,anom} + \mu_i [D\bar{J}_{anom}^i - \bar{\mathfrak{A}}^i] = D\bar{q}_{anom} + a \wedge \bar{q}_{anom} + \bar{J}_{anom}^i \wedge E_i \quad (\text{A.1})$$

with  $\bar{\mathfrak{A}}^i \equiv t^i e^{t.B} \wedge u \wedge (t.E)$  in the form

$$\begin{aligned} \bar{q}_{anom} &= - \left[ \frac{(2t.\mu \omega - 1)e^{t.(B+2\mu\omega)} + e^{t.B}}{4\omega^2} \right] \wedge u \\ \bar{J}_{anom}^i &= -t^i \left[ \frac{e^{t.(B+2\mu\omega)} - e^{t.B}}{2\omega} \right] \wedge u \\ \bar{J}_{S,anom} &= 0 \end{aligned} \quad (\text{A.2})$$

In this appendix, we will examine the uniqueness of this solution. Since we are solving an inhomogeneous linear equation, the difference between various solutions of this equation satisfies the *homogeneous* equation

$$TD\bar{J}_{S,rev} + \mu_i D\bar{J}_{rev}^i = D\bar{q}_{rev} + a \wedge \bar{q}_{rev} + \bar{J}_{rev}^i \wedge E_i \quad (\text{A.3})$$

We have used the suffix *rev* to denote that physically these denote transport processes which do not result in entropy production and hence are reversible. Evidently, it is too ambitious to try to classify *all* possible reversible transport processes which can happen in hydrodynamics, so we will content ourselves with describing some classes of solutions to the above equations which are close in form to the transport due to global anomalies.

In examining the solution-space of the equation (A.1), it is insightful to first prove the following theorem

**Theorem.** Consider a  $\mathcal{V}_{anom}$  of the form

$$\mathcal{V}_{anom} \equiv - \frac{(\mathfrak{f}[B_i + 2\mu_i\omega] - \mathfrak{f}[B_i])}{2\omega} - T^2\omega \wedge \mathfrak{g}[B_i + 2\mu_i\omega, T\omega] \quad (\text{A.4})$$

where  $\mathfrak{f}$  and  $\mathfrak{g}$  are some Taylor-expandable functions<sup>16</sup> at the origin (i.e., are analytic at the origin). Construct from this the following currents

$$\begin{aligned}
\bar{\mathcal{G}}_{rev} &= -\frac{1}{2\omega} [\mathcal{V}_{anom} - \mathcal{V}_{anom}(\omega = 0)] \wedge u \\
\bar{q}_{rev} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u = \bar{\mathcal{G}}_{rev} + T \bar{J}_{S,rev} + \mu_i \bar{J}_{rev}^i \\
\bar{J}_{rev}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u = -\frac{\partial \bar{\mathcal{G}}_{rev}}{\partial \mu_i} \\
\bar{J}_{S,rev} &= -\frac{\partial \bar{\mathcal{G}}_{rev}}{\partial T}
\end{aligned} \tag{A.5}$$

Then they satisfy

$$TD \bar{J}_{S,rev} + \mu_i \left( D \bar{J}_{rev}^i - \frac{\delta^2 \mathfrak{f}[B_k]}{\delta B_i \delta B_j} \wedge u \wedge E_j \right) = D \bar{q}_{rev} + a \wedge \bar{q}_{rev} + \bar{J}_{rev}^i \wedge E_i \tag{A.6}$$

Notice that the main solution of this paper appears as a special case of this general theorem (with  $\mathfrak{f}[B_i] = e^{t \cdot B}$  and  $\mathfrak{g} = 0$ ). At least in this subspace of solutions this is the unique solution at zero temperature.

It can be checked by explicit computation that these class of solutions satisfy first-law type relations

$$\begin{aligned}
\frac{\partial \bar{q}_{anom}}{\partial T} &= T \frac{\partial \bar{J}_{S,rev}}{\partial T} + \mu_k \frac{\partial \bar{J}_{rev}^k}{\partial T} \\
\frac{\partial \bar{q}_{anom}}{\partial \mu_i} &= T \frac{\partial \bar{J}_{S,rev}}{\partial \mu_i} + \mu_k \frac{\partial \bar{J}_{rev}^k}{\partial \mu_i}
\end{aligned} \tag{A.7}$$

and a generalized Onsager- type Reciprocity relation

$$\frac{\delta \bar{q}_{rev}}{\delta B_i} = \frac{1}{2} \frac{\delta \bar{J}_{rev}^i}{\delta \omega} \tag{A.8}$$

Now, it is clear that if we take  $\mathfrak{f} = 0$ , we get a large class of homogeneous solutions that we want. In  $2n$  dimensions,  $\mathfrak{g}$  takes the general form

$$\mathfrak{g} = \sum_{k=0}^{n-1} \frac{(T\omega)^{n-1-k}}{k!(n-1-k)!} \wedge \mathfrak{g}_k^{i_1 \dots i_k} (B + 2\mu\omega)_{i_1} \wedge \dots \wedge (B + 2\mu\omega)_{i_k} \tag{A.9}$$

---

<sup>16</sup>If we only want the parts relevant to  $2n$  spacetime dimensions, we can take treat  $\mathfrak{f}$  and  $\mathfrak{g}$  as  $(n+1)$ th and  $(n-1)$ th degree homogeneous polynomials of their arguments respectively. This makes  $\mathcal{V}_{anom}$  into a formal spatial  $2n$  form.

where  $\mathfrak{g}_k^{i_1 i_2 \dots i_k}$  is some invariant tensor of the global symmetry<sup>17</sup>.

Note that this construction which we just outlined is a generalization of the finite temperature corrections in (3+1)d proposed by the authors of [8]. In (3+1)d, for example,  $n = 2$ , we can take

$$\mathfrak{g} = \mathfrak{g}_0(T\omega) + \mathfrak{g}_1^i(B + 2\mu\omega)_i \quad (\text{A.11})$$

which gives

$$\begin{aligned} \bar{\mathcal{G}}_{rev} &= \left[ \frac{1}{2} T^2 \mathfrak{g}_1^i B_i + \left( \frac{1}{2} T^3 \mathfrak{g}_0 + T^2 \mu_i \mathfrak{g}_1^i \right) \omega \right] \wedge u \\ \bar{q}_{rev} &= -T^2 \left[ \frac{1}{2} \mathfrak{g}_1^i B_i + (T \mathfrak{g}_0 + 2 \mathfrak{g}_1^i \mu_i) \omega \right] \wedge u \\ \bar{J}_{rev}^i &= -T^2 \mathfrak{g}_1^i \omega \wedge u \\ \bar{J}_{S,rev} &= -T \left[ \mathfrak{g}_1^i B_i + \left( \frac{3}{2} T \mathfrak{g}_0 + 2 \mathfrak{g}_1^i \mu_i \right) \omega \right] \wedge u \end{aligned} \quad (\text{A.12})$$

and these exactly correspond to the finite temperature corrections proposed in [8], provided we shift to the Landau frame using the procedure described in section §4.

The (3+1)d free fermion calculation<sup>18</sup> in [10] gives  $\mathfrak{g}_1^i \propto \text{tr } T^i$ , i.e.,  $\mathfrak{g}_1^i$  is non-zero in free fermion theories if and only if there is a mixed gravitational anomaly. It is unknown what happens in interacting theories. As far as the author is aware  $\mathfrak{g}_0$  has never been calculated in free theory or otherwise. Given that the associated transport violates CP, it might be of some phenomenological relevance as a possible measure of CP violation (say in the early universe).

A similar exercise can be repeated in arbitrary dimensions to get the finite temperature corrections from the equations (A.5) and (A.9).

## B. Explicit expressions in various cases

In this appendix, we present the explicit solutions for  $B = 0$  and  $\omega = 0$  followed by the explicit solutions for all even dimensions till d=10.

---

<sup>17</sup>Curiously, such invariant tensors inevitably occur for odd  $n - k$  (only odd  $n - k$  occur if  $\mathfrak{g}$  is taken to be an even function of  $T\omega$ ) whenever there is a mixed global-gravitational anomaly of the form

$$D\bar{J}^{i_1} \supset \sum_{k=1}^n \frac{\alpha_k}{k!(n-k+1)!} \text{Tr}(R^{n-k+1}) \wedge \mathfrak{g}_k^{i_1 \dots i_k} F_{i_2} \wedge \dots \wedge F_{i_k} \quad (\text{A.10})$$

hence it is interesting to speculate that theories with mixed anomalies generically have this kind of transport - in fact, something like this happens for free fermions in (3+1)dimensions [10] but the question of whether such a relation continues to hold for strongly coupled theories with same coefficient is open.

More generally, there is the question of finding the transport consistent with the second law in the presence of mixed anomalies which is yet to be answered.

<sup>18</sup>Also see old calculations by Vilenkin in the context of neutrinos [11, 12, 13, 14, 15, 16].

All these follow from our solution - keeping only parts relevant to  $d = 2n$  dimensions

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= t^i \frac{(t.F)^n}{n!} \\
\mathcal{V}_{anom} &= - \frac{[t.(B + 2\mu\omega)]^{n+1} - [t.B]^{n+1}}{(2\omega)(n+1)!} \\
\bar{\mathcal{G}}_{anom} &= \frac{[t.(B + 2\mu\omega)]^{n+1} - [t.B]^{n+1} - (n+1)[2t.\mu\omega][t.B]^n}{(2\omega)^2(n+1)!} \\
\bar{q}_{anom} &= - \left[ \frac{(n+1)[2t.\mu\omega][t.(B + 2\mu\omega)]^n - [t.(B + 2\mu\omega)]^{n+1} + [t.B]^{n+1}}{(2\omega)^2(n+1)!} \right] \wedge u \\
\bar{J}_{anom}^i &= -t^i \left[ \frac{[t.(B + 2\mu\omega)]^n - [t.B]^n}{(2\omega)n!} \right] \wedge u \\
\bar{J}_{S,anom} &= 0
\end{aligned} \tag{B.1}$$

### **B=0 case : Chiral Vortical Effect**

We now present the answer in arbitrary dimensions when the gauge fields are turned off

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= 0 \\
\bar{\mathcal{G}}_{anom} &= \frac{1}{(n+1)!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_0} \mu_{i_1} \mu_{i_2} \dots \mu_{i_n} (2\omega)^{n-1} \wedge u \\
\bar{q}_{anom} &= - \frac{n}{(n+1)!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_0} \mu_{i_1} \mu_{i_2} \dots \mu_{i_n} (2\omega)^{n-1} \wedge u \\
\bar{J}_{anom}^{i_0} &= - \frac{1}{n!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_1} \mu_{i_2} \dots \mu_{i_n} (2\omega)^{n-1} \wedge u \\
\bar{J}_{S,anom} &= 0
\end{aligned} \tag{B.2}$$

This gives the chiral vortical effect in arbitrary dimensions due to the anomaly. Note that it is still present when the anomaly itself has been turned off.

### **$\omega=0$ case : Chiral Magnetic effect**

We now present the answer in arbitrary dimensions when the vorticity is turned off

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= t^i \frac{(t.F)^n}{n!} \\
\bar{\mathcal{G}}_{anom} &= \frac{1}{2!(n-1)!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_0} \mu_{i_1} B_{i_2} B_{i_3} \dots B_{i_n} \wedge u \\
\bar{q}_{anom} &= - \frac{1}{2!(n-1)!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_0} \mu_{i_1} B_{i_2} B_{i_3} \dots B_{i_n} \wedge u \\
\bar{J}_{anom}^i &= - \frac{1}{(n-1)!} \mathfrak{C}^{i_0 i_1 i_2 \dots i_n} \mu_{i_1} B_{i_2} B_{i_3} \dots B_{i_n} \wedge u \\
\bar{J}_{S,anom} &= 0
\end{aligned} \tag{B.3}$$

This gives the chiral magnetic effect in arbitrary dimensions due to the anomaly.

We now present the explicit solutions for all even dimensions till  $d=10$  for ready reference.

**d=2**

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \mathfrak{E}^{ij} F_j \\
\bar{\mathcal{G}}_{anom} &= \frac{1}{2!} \mathfrak{E}^{jk} \mu_{jk}^2 (2\omega) \wedge u \\
\mathcal{V}_{anom} &= -\mathfrak{E}^{jk} [\mu_j B_k + \mu_{jk}^2 \omega] \\
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u = -\frac{1}{2} \mathfrak{E}^{ij} \mu_{ij}^2 u \\
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u = -\mathfrak{E}^{ij} \mu_j u \\
\bar{J}_{S,anom} &= \frac{1}{2\omega} \frac{\partial \mathcal{V}_{anom}}{\partial T} \wedge u = 0
\end{aligned} \tag{B.4}$$

**d=4**

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \frac{1}{2!} \mathfrak{E}^{ijk} F_{jk}^2 \\
\bar{\mathcal{G}}_{anom} &= \left[ \frac{1}{3!} \mathfrak{E}^{ijk} \mu_{ijk}^3 (2\omega) + \frac{1}{2!} \mathfrak{E}^{ijk} \mu_{ij}^3 B_k \right] \wedge u \\
\mathcal{V}_{anom} &= -\mathfrak{E}^{jkl} \mu_j \left[ \frac{1}{2} B_{kl}^2 + B_k \wedge \mu_l \omega + \frac{2}{3} \mu_{kl}^2 \omega^2 \right] \wedge u \\
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u = -\mathfrak{E}^{ijk} \mu_{ij}^2 \left[ \frac{1}{2} B_k + \frac{2}{3} \mu_k \omega \right] \wedge u \\
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u = -\mathfrak{E}^{ijk} \mu_j [B_k + \mu_k \omega] \wedge u \\
\bar{J}_{S,anom} &= \frac{1}{2\omega} \frac{\partial \mathcal{V}_{anom}}{\partial T} \wedge u = 0
\end{aligned} \tag{B.5}$$

**d=6**

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \frac{1}{3!} \mathfrak{E}^{ijkl} F_{jkl}^3 \\
\bar{\mathcal{G}}_{anom} &= \left[ \frac{1}{4!} \mathfrak{E}^{ijkl} \mu_{ijkl}^4 (2\omega)^2 + \frac{1}{3!} \mathfrak{E}^{ijkl} \mu_{ijk}^3 (2\omega) B_l + \frac{1}{2!2!} \mathfrak{E}^{ijkl} \mu_{ij}^2 B_{kl}^2 \right] \wedge u \\
\mathcal{V}_{anom} &= -\mathfrak{E}^{jklm} \mu_j \left[ \frac{1}{6} B_{klm}^3 + \frac{1}{2} B_{kl}^2 \wedge \mu_m \omega \right. \\
&\quad \left. + \frac{2}{3} B_k \wedge \mu_{lm}^2 \omega^2 + \frac{1}{3} \mu_{klm}^3 \omega^3 \right] \wedge u \\
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u \\
&= -\mathfrak{E}^{ijkl} \mu_{ij}^2 \left[ \frac{1}{4} B_{kl}^2 + \frac{2}{3} B_k \wedge \mu_l \omega + \frac{1}{2} \mu_{kl}^2 \omega^2 \right] \wedge u \\
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u \\
&= -\mathfrak{E}^{ijkl} \mu_j \left[ \frac{1}{2} B_{kl}^2 + B_k \wedge \mu_l \omega + \frac{2}{3} \mu_{kl}^2 \omega^2 \right] \wedge u \\
\bar{J}_{S,anom} &= \frac{1}{2\omega} \frac{\partial \mathcal{V}_{anom}}{\partial T} \wedge u = 0
\end{aligned} \tag{B.6}$$



**d=8**

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \frac{1}{4!} \mathfrak{C}^{ijklm} F_{jklm}^4 \\
\mathcal{V}_{anom} &= -\mathfrak{C}^{jklmn} \mu_j \left[ \frac{1}{24} B_{klmn}^4 + \frac{1}{6} B_{klm}^3 \wedge \mu_n \omega \right. \\
&\quad \left. + \frac{1}{3} B_{kl}^2 \wedge \mu_{mn}^2 \omega^2 + \frac{1}{3} B_k \wedge \mu_{lmn}^3 \omega^3 + \frac{2}{15} \mu_{klmn}^4 \omega^4 \right] \wedge u \\
\bar{\mathcal{G}}_{anom} &= \left[ \frac{1}{5!} \mathfrak{C}^{ijklm} \mu_{ijklm}^5 (2\omega)^3 + \frac{1}{4!} \mathfrak{C}^{ijklm} \mu_{ijkl}^4 (2\omega)^2 B_m \right. \\
&\quad \left. + \frac{1}{3!2!} \mathfrak{C}^{ijklm} \mu_{ijk}^3 (2\omega) B_{lm}^2 + \frac{1}{2!3!} \mathfrak{C}^{ijklm} \mu_{ij}^2 B_{klm}^3 \right] \wedge u \\
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u \\
&= -\mathfrak{C}^{ijklm} \mu_{ij}^2 \left[ \frac{1}{12} B_{klm}^3 + \frac{1}{3} B_{kl}^2 \wedge \mu_m \omega \right. \\
&\quad \left. + \frac{1}{2} B_k \wedge \mu_{lm}^2 \omega^2 + \frac{4}{15} \mu_{klm}^3 \omega^3 \right] \wedge u \\
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u \\
&= -\mathfrak{C}^{ijklm} \mu_j \left[ \frac{1}{6} B_{klm}^3 + \frac{1}{2} B_{kl}^2 \wedge \mu_m \omega \right. \\
&\quad \left. + \frac{2}{3} B_k \wedge \mu_{lm}^2 \omega^2 + \frac{1}{3} \mu_{klm}^3 \omega^3 \right] \wedge u \\
\bar{J}_{S,anom} &= \frac{1}{2\omega} \frac{\partial \mathcal{V}_{anom}}{\partial T} \wedge u = 0
\end{aligned} \tag{B.7}$$

**d=10**

$$\begin{aligned}
\bar{\mathfrak{A}}^i &= \frac{1}{5!} \mathfrak{C}^{ijklmn} F_{jklmn}^5 \\
\mathcal{V}_{anom} &= -\mathfrak{C}^{jklmnp} \mu_j \left[ \frac{1}{120} B_{klmnp}^5 + \frac{1}{24} B_{klmn}^4 \wedge \mu_p \omega + \frac{1}{9} B_{klm}^3 \wedge \mu_{np}^2 \omega^2 \right. \\
&\quad \left. + \frac{1}{6} B_{kl}^2 \wedge \mu_{mnp}^3 \omega^3 + \frac{2}{15} B_k \mu_{lmnp}^4 \omega^4 + \frac{2}{45} \mu_{klmnp}^5 \omega^5 \right] \\
\bar{\mathcal{G}}_{anom} &= \left[ \frac{1}{6!} \mathfrak{C}^{ijklmn} \mu_{ijklmn}^6 (2\omega)^4 + \frac{1}{5!} \mathfrak{C}^{ijklmn} \mu_{ijklm}^5 (2\omega)^3 B_n \right. \\
&\quad + \frac{1}{4!2!} \mathfrak{C}^{ijklmn} \mu_{ijklm}^4 (2\omega)^2 B_{mn}^2 + \frac{1}{3!3!} \mathfrak{C}^{ijklmn} \mu_{ijk}^3 (2\omega) B_{lmn}^3 \\
&\quad \left. + \frac{1}{2!4!} \mathfrak{C}^{ijklmn} \mu_{ij}^2 B_{klmn}^4 \right] \wedge u \\
\bar{q}_{anom} &= \frac{1}{2} \frac{\delta \mathcal{V}_{anom}}{\delta \omega} \wedge u = -\mathfrak{C}^{ijklmn} \mu_i \mu_j \left[ \frac{1}{48} B_{klmn}^4 + \frac{1}{9} B_{klm}^3 \wedge \mu_n \omega \right. \\
&\quad \left. + \frac{1}{4} B_{kl}^2 \wedge \mu_{mn}^2 \omega^2 + \frac{4}{15} B_k \wedge \mu_{lmn}^3 \omega^3 + \frac{1}{9} \mu_{klmn}^4 \omega^4 \right] \wedge u
\end{aligned}$$

$$\begin{aligned}
\bar{J}_{anom}^i &= \frac{\delta \mathcal{V}_{anom}}{\delta B_i} \wedge u = -\epsilon^{ijklmn} \mu_j \left[ \frac{1}{24} B_{klmn}^4 + \frac{1}{6} B_{klm}^3 \wedge \mu_n \omega \right. \\
&\quad \left. + \frac{1}{3} B_{kl}^2 \wedge \mu_{mn}^2 \omega^2 + \frac{1}{3} B_k \wedge \mu_{lmn}^3 \omega^3 + \frac{2}{15} \mu_{klmn}^4 \omega^4 \right] \wedge u \\
\bar{J}_{S,anom} &= \frac{1}{2\omega} \frac{\partial \mathcal{V}_{anom}}{\partial T} \wedge u = 0
\end{aligned} \tag{B.8}$$

### C. Notation

We work in the  $(-++\dots)$  signature. The dimensions of the spacetime in which the fluid lives is denoted by  $d = 2n$ . The Greek indices  $\mu, \nu = 0, 1, \dots, d-1$  are used as space-time indices, whereas the Latin indices  $i, j, k \dots$  are used as the flavor charge indices. .

We denote Hodge-duals by an overbar - for example,  $\bar{J}_i$  is the  $2n-1$  form Hodge-dual to the 1-form  $J_i$ . We mostly just use the Hodge-duality between 1-forms and  $2n-1$  forms and our conventions are completely defined by the following statement- given any  $2n-1$  form  $\bar{V}$  hodge-dual to  $V_\mu$  and a 1-form  $A_\mu$ , we have

$$\begin{aligned}
D\bar{V} &= (D_\mu V^\mu) \text{Vol}_{2n} \\
A \wedge \bar{V} &= -\bar{V} \wedge A = A_\mu V^\mu \text{Vol}_{2n}
\end{aligned} \tag{C.1}$$

Given a 0-form  $\alpha$  its Hodge-dual  $2n$ -form is simply  $\bar{\alpha} \equiv \alpha \text{Vol}_{2n}$ .

We have included a table with other useful parameters used in the text. In the table C, the relevant equations are denoted by their respective equation numbers appearing inside parentheses.

### References

- [1] P. Kovtun, D. T. Son, and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601, [[hep-th/0405231](#)].
- [2] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045, [[arXiv:0712.2456](#)].
- [3] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, *JHEP* **09** (2008) 054, [[arXiv:0708.1770](#)].
- [4] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055, [[arXiv:0809.2488](#)].
- [5] N. Banerjee *et al.*, *Hydrodynamics from charged black branes*, *JHEP* **01** (2011) 094, [[arXiv:0809.2596](#)].
- [6] M. Torabian and H.-U. Yee, *Holographic nonlinear hydrodynamics from AdS/CFT with multiple/non-Abelian symmetries*, *JHEP* **08** (2009) 020, [[arXiv:0903.4894](#)].
- [7] D. T. Son and P. Surowka, *Hydrodynamics with Triangle Anomalies*, *Phys. Rev. Lett.* **103** (2009) 191601, [[arXiv:0906.5044](#)].
- [8] Y. Neiman and Y. Oz, *Relativistic Hydrodynamics with General Anomalous Charges*, *JHEP* **03** (2011) 023, [[arXiv:1011.5107](#)].

Table of Notation			
Symbol	Definition	Symbol	Definition
$u^\mu, u$	Fluid velocity, 1-form	$P_{\mu\nu}$	$g_{\mu\nu} + u_\mu u_\nu$
$g_{\mu\nu}$	Spacetime metric	$p$	Fluid pressure
$\varepsilon$	Fluid energy density	$s$	Fluid entropy density
$n^i$	Fluid charge density	$T$	Fluid temperature
$\mu_i$	Chemical potentials	$J^{i\mu}$	Charge currents with anomalies
$T^{\mu\nu}$	Energy-momentum tensor of the fluid	$\mathfrak{C}^{ij\dots}$	Anomaly coefficient
$J_S^\mu$	Entropy current	$\bar{q}_{anom}$	Hodge-dual of $q_{anom}^\mu$ $2n - 1$ form
$q_{anom}^\mu$	Anomaly-induced heat current	$\bar{J}_{anom}^i$	Hodge-dual of $J_{anom}^{i\mu}$ $2n - 1$ form
$J_{anom}^{i\mu}$	Anomaly-induced Charge current	$\bar{J}_{S,anom}$	Hodge-dual of $J_{S,anom}^\mu$ $2n - 1$ form
$J_{S,anom}^\mu$	Anomaly-induced Entropy current	$E_i^\mu, E_i$	Rest frame electric field $F_{i\mu\nu}u^\nu$ , 1-form
$F_{i\mu\nu}, F_i$	non-dynamical gauge field strength, 2-form	$\mathfrak{A}^i$	Anomaly in the $i$ th current $D\bar{J}^i \equiv \mathfrak{A}^i$
$B_{i\mu\nu}, B_i$	Rest frame magnetic fields $F_i - u \wedge E_i$	$D_\mu$	Flavor/Lorentz Covariant derivative
$a_\mu, a$	Acceleration field $(u.D)u_\mu$ , 1-form	$\omega_{\mu\nu}, \omega$	Fluid vorticity, 2-form
$\sigma_{\mu\nu}$	Shear strain rate	$\bar{\mathcal{G}}_{anom}$	Anomaly-induced the Gibbs current (Hodge dual)
$\mathcal{V}_{anom}$	A formal spatial $2n$ -form encoding anomalous transport		

- [9] I. Amado, K. Landsteiner, and F. Pena-Benitez, *Anomalous transport coefficients from Kubo formulas in Holography*, [arXiv:1102.4577](#).
- [10] K. Landsteiner, E. Megias, and F. Pena-Benitez, *Gravitational Anomaly and Transport*, [arXiv:1103.5006](#).
- [11] A. Vilenkin, *Parity violating currents in thermal radiation*, *Phys. Lett.* **B80** (1978) 150–152.
- [12] A. Vilenkin, *Macroscopic parity violating effects: neutrino fluxes from rotating black holes and in rotating thermal radiation*, *Phys. Rev.* **D20** (1979) 1807–1812.
- [13] A. Vilenkin, *Equilibrium parity violating current in a magnetic field*, *Phys. Rev.* **D22** (1980) 3080–3084.
- [14] A. Vilenkin, *Quantum field theory at finite temperature in a rotating system*, *Phys. Rev.* **D21** (1980) 2260–2269.
- [15] A. Vilenkin, *Cancellation of equilibrium parity violating currents*, *Phys. Rev.* **D22** (1980) 3067–3079.
- [16] A. Vilenkin, *Parity nonconservation and neutrino transport in magnetic fields*, *Astrophys. J.* **451** (1995) 700–702.

- [17] D. E. Kharzeev and H.-U. Yee, *Anomalies and time reversal invariance in relativistic hydrodynamics: the second order and higher dimensional formulations*, [arXiv:1105.6360](#).
- [18] W. A. Bardeen and B. Zumino, *Consistent and Covariant Anomalies in Gauge and Gravitational Theories*, *Nucl.Phys.* **B244** (1984) 421. Revised version.
- [19] C.-S. Chu, P.-M. Ho, and B. Zumino, *Non-Abelian Anomalies and Effective Actions for a Homogeneous Space  $G/H$* , *Nucl. Phys.* **B475** (1996) 484–504, [[hep-th/9602093](#)].
- [20] R. Loganayagam and P. Surowka, *To appear*, .